On Partitioning of Hypergraphs

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Abstract

We study Edge-Isoperimetric Problems (EIP) for hypergraphs and extend some technique in this area from graphs to hypergraphs. In particular, we establish some new results on a relationship between the EIP and some extremal poset problems, and apply them to obtain an exact solution of the EIP for certain hypergraph families. We also show how to solve the EIP on hypergraphs in some cases when the link to posets does not work. Another outcome of our results is a new series of hypergraphs admitting nested solutions in the EIP.

1 Problem statement and motivation

Let $H = \{V_H, E_H\}$ be a hypergraph where V_H is the the vertex set and $E_H \subseteq 2^{V_H}$ is the set of hyperedges. For $A \subseteq V_H$ denote $\overline{A} = V_H \setminus A$ and

$$\begin{aligned} \theta_H(A) &= \{e \in E_H \mid e \cap A \neq \emptyset \text{ and } e \cap \overline{A} \neq \emptyset\} \\ \theta_H(m) &= \min_{\substack{A \subseteq V_H \\ |A|=m}} |\theta_H(A)|. \end{aligned}$$

In other words, $\theta_H(A)$ is the set of hyperedges connecting A with its complement in V_H, or, what is commonly said, $\theta_H(A)$ is the hyperedge-cut. The Edge-Isoperimetric Problem (EIP) on H consists of finding for a given m, $1 \leq m \leq |V_H|$, a set $A \subseteq V_G$ such that $|A| = m$ and $|\theta_H(A)| = \theta_H(m)$. This problem is known to be NP-complete even for graphs.

We mostly will be dealing with another version of the EIP, which also makes sense for practical applications. Instead of minimizing the number of broken connections we maximize the number of ones being localized within a board. For $A \subseteq V_H$ denote

$$\begin{aligned} E_H(A) &= \{e \in E_H \mid e \subseteq A\}, \\ E_H(m) &= \max_{\substack{A \subseteq V_H \\ |A|=m}} |E_H(A)|, \\ \delta_H(m) &= E_H(m+1) - E_H(m) \quad \text{for } m = 0, \ldots, |V_H|-1. \end{aligned}$$

The hyperedges in $E_H(A)$ are called *inner hyperedges* (with respect to A). We call a set $A \subseteq V_H$ θ-*optimal* (resp. E-optimal) if $|\theta_H(A)| = \theta_H(|A|)$ (resp. $|E_H(A)| = E_H(|A|)$). Our objective is to compute the functions $\theta_H(m)$ and $E_H(m)$ for $m = 1, 2, \ldots, |V_H|$ and construct the corresponding optimal sets.

The second version of the EIP is closely related to the first one. This relationship is provided by the following partition of E_H

$$E_H = E_H(A) \cup E_H(\overline{A}) \cup \theta_H(A), \tag{1}$$

which implies a lower bound

$$\theta_H(m) \geq |E_H| - E_H(m) - E_H(|V_H| - m). \tag{2}$$

Note that the both versions of the EIP considered on graphs are equivalent for regular graphs. If a graph G is k-regular, then counting the sum s of the vertex degrees for the set A provides $s = k|A|$. On the other hand each inner edge is counted exactly twice, so

$$k|A| = 2|E_G(A)| + |\theta_G(A)|. \tag{3}$$

Hence, maximization of $|E_G(A)|$ for regular graphs is equivalent to minimization of $|\theta_G(A)|$. For non-regular graphs and even for regular hypergraphs the identity (3) does not hold, in general, and the two versions of the EIP can be essentially different. Anyway, the new technique we are developing in this paper can be applied, under certain conditions, to solving both versions of the EIP even for some irregular hypergraphs.

We are particularly interested in the case when the EIP admits *nested solutions*, i.e., when there exists a total order on V_G, such that any initial segment of this order constitutes an E-optimal (resp. θ-optimal) set. We call such order E-*order* (resp. θ-*order*).

Furthermore, we focus on the EIP on the cartesian products of hypergraphs. Given hypergraphs $H' = \{V_{H'}, E_{H'}\}$ and $H'' = \{V_{H''}, E_{H''}\}$, the cartesian product $H' \times H''$ is defined as the hypergraph with the vertex set $V_{H'} \times V_{H''}$ where two vertices (u', v') and (u'', v'') belong to the same hyperedge iff either $u' = u''$ and $\{v', v''\} \subseteq e'$ for some $e' \in E_{H'}$, or $v' = v''$ and $\{u', u''\} \subseteq e''$ for some $e'' \in E_{H''}$. It is easily seen that $|V_{H' \times H''}| = |V_{H'}| \cdot |V_{H''}|$ and $|E_{H' \times H''}| = |V_{H'}| \cdot |E_{H''}| + |V_{H''}| \cdot |E_{H'}|$. Denote by H^n the n-th cartesian power of a hypergraph H, i.e. $H^n = H \times \cdots \times H$ (n times).

The EIP and related problems arise in many practical areas. For example, hypergraph partitioning has emerged as a central issue in VLSI design [1, 13]. In VLSI placement, a divide and conquer approach is taken where the circuit is hierarchically divided into smaller components by using hypergraph partitioning [18]. The EIP also finds applications in *rapid prototyping* where the goal typically is to partition a circuit into a minimal number of FPGAs (Field Programmable Gate Arrays) under different constraints such as available number of pins and routing resources [10, 14, 15, 16, 23]. Another application area is the *design for testability* of VLSI circuits where the circuit is partitioned into smaller parts to facilitate testing [22]. However the importance of the hypergraph partitioning problem goes beyond the VLSI design. For example, electrical circuits with multiple-pin nets are readily modeled as hypergraphs. Other applications include data mining [19], efficient storage of large databases on disks [20], clustering and partitioning of the roadmap database for routing applications [21], de-clustering data in parallel databases [20]. In the last years one has observed a blossoming of graph and hypergraph partitioning algorithms and software packages (METIS, MELO, Paraboli, SCOTCH). The solution quality and reliability improvements which have come along is remarkable and would have been unpredictable only a few years ago [1, 13, 11, 12]. We refer to [1] for a recent survey of the problem and for detailed references. In [3] we analyze the performance of greedy strategies in the context of hypergraph bisection by extending to hypergraphs the successful performance of heuristic schemes previously obtained on graphs [2, 4]. In the work related to heuristics, the availability of a set of instances with known optimal solution is of a high importance because it allows to benchmark the performance of the heuristic techniques relative to the optimal solution, therefore obtaining useful knowledge about the algorithm performance.

The paper is organized as follows. In the next section we will show that the EIP on hypergraphs is closely related to some extremal problems on partially ordered sets (posets). A similar relationship is also known for graphs [6]. In fact, for any graph one can construct a representing poset, in a sense. However, it is known that not every poset represents some graph. A new contribution to the theory is Theorem 1, where we show that every poset satisfying a necessary condition represents some hypergraph. Exploring this relationship, we present in Section 3 several examples of application of the poset technique to the EIP on some hypergraph families. In Section 4 we adopt the known compression technique for graphs to hypergraphs to be used in Sections 5 and 6. Section 5 is devoted to the situation when a hypergraph does not satisfy the necessary condition for the poset representation, so our poset approach from Section 2 won't work. In Theorem 3 we present a solution to the EIP for a special hypergraph family of that kind. In Section 6 we present two types of hypergraphs, for which both versions of the EIP are equivalent. Conclusions and open problems in Section 7 complete the paper.

2 Representation of hypergraphs by posets

Following [6], we establish a relationship between the EIP on hypergraphs and some extremal problems on partially ordered sets (posets). Let $(P, \leq_P)$ be a poset with a partial order $\leq_P$. The poset $(P, \leq_P)$ is called *ranked* if there exists a *rank function* $r_P : P \mapsto \mathbf{R}^{\geq 0}$ such that $r(x) = 0$ for some minimal element of P (in the partial order $\leq_P$), and $r_P(x) = r_P(y) - 1$ whenever $x <_P y$ and there is no $z \in P$ with $x <_P z <_P y$.

A set $I \subseteq P$ is called *ideal* if whenever $x \in I$ and $y \leq_P x$ one has $y \in I$. For an ideal

$I \subseteq P$ denote

$$\begin{aligned} R_P(I) &= \sum_{x\in I} r_P(x), \\ R_P(m) &= \max_{I\subseteq P,\ |I|=m} R_P(I), \text{ for } m = 1,\dots,|P| \\ \omega_P(m) &= R_P(m+1) - R_P(m), \text{ for } m = 0,\dots,|P|-1. \end{aligned}$$

The *Maximum Rank Ideal* (MRI) problem on the poset $(P, \leq_P)$ consists of finding for a given m, $1 \leq m \leq |P|$, an ideal $I^* \subseteq P$, such that $|I^*| = m$ and $R_P(I^*) = R_P(m)$. We call such ideals optimal and say that P admits an MRI-order if there exists a total order on P whose any initial segment constitutes an optimal ideal.

For a given hypergraph H, we say that H is *represented* by a ranked poset $(P, \leq_P)$ with $|V_H| = |P|$ if $E_H(m) = R_P(m)$ for $m = 1,\dots,|P|$. A hypergraph might be represented by several non-isomorphic ranked posets and a ranked poset might represent several non-isomorphic hypergraphs.

Lemma 1 *A hypergraph H such that*

$$\delta_H(m) - \delta_H(m-1) \leq 1 \quad \textit{for} \quad m = 1,\dots,|V_H|-1, \tag{4}$$

can be represented by a ranked poset admitting an MRI-order.

Proof.
Denote $p = |V_H|$. We show by induction on p that for any sequence $\delta_H(0),\dots,\delta_H(p-1)$ satisfying (4) there exists a ranked poset for which $\omega_P(i) = \delta_H(i)$ for $i = 0,\dots,p-1$. We apply an algorithm from [6] to construct such a poset.

The statement is, obviously, true for $p = 1$, so assume $p \geq 2$. Let $(P, \leq_P)$ be a ranked poset for the sequence $\delta_H(0),\dots,\delta_H(p-2)$ constructed by induction. Now, we extend P by adding to it a new element x at level $\delta_H(p-1)$ and extend the partial order $\leq_P$ by setting x to be greater than any element of P at level $\delta_H(p-1)-1$ (if such exist). The correctness of this construction is guaranteed by condition (4). Obviously, the obtained poset is ranked.

The algorithm naturally provides a total order $\mathcal{O}$ of the poset elements as they appear in the construction. We show by induction on p that the order $\mathcal{O}$ is an MRI-order. This is evident for $p = 1$. For $p > 1$ let $I \subseteq P$ be an optimal ideal. Denote by $(P', \leq_{P'})$ the subposet defined by the first $p-1$ elements in order $\mathcal{O}$. Our goal will be achieved if we show that for $|I| < |P|$ there exists an ideal $I' \subseteq P'$ with $R_P(I') \geq R_P(I)$. Indeed, if the element p is not in I, this follows from the induction hypothesis. Otherwise, denote $I'' = I \setminus \{p\}$. Since I is an ideal, by the construction of P, all elements of P of rank $r_P(p) - 1$ or less are in I. Since $|I| < |P|$, one has $|I''| < |P'|$. Let I' be an ideal obtained from I'' by adding to it some element $z \in P'$. Obviously, the element z does exist and $r_P(z) \geq r_P(p)$. But then

$$R_P(I') = R_{P'}(I'') + r_{P'}(z) = R_P(I'') + r_P(z) = R_P(I) - r_P(p) + r_P(z) \geq R_P(I).$$

Hence, the order $\mathcal{O}$ is an MRI-order. The condition $\omega_P(i) = \delta_H(i)$ for $i = 0, \ldots, p-1$, follows from the construction. Since $E_H(m) = \sum_{i=0}^{m-1} \delta_H(i)$ and $R_P(m) = \sum_{i=0}^{m-1} \omega_P(i)$, the poset $(P, \leq_P)$ represents H. □

An analog of Lemma 1 is known for graphs [6]. The condition (4) is automatically satisfied if a graph admits an E-order. For hypergraphs the picture is different. The condition (4) might not be satisfied even if a hypergraph admits an E-order (see Section 5 for an example). On the other hand, it is known [6] that not every ranked poset represents a graph, no matter if the poset admits an MRI-order or not. An example of such poset is shown in Fig. 1(a). It is interesting, that every ranked poset $(P, \leq_P)$ admitting an MRI-order represents some hypergraph.

We construct a hypergraph $H(P)$ represented by the poset $(P, \leq_P)$ by induction on $p = P$. For $p = 1$ the hypergraph is trivial, so assume $p \geq 2$. Since $(P, \leq_P)$ admits an MRI-order, there exists a labeling of P by numbers $0, 1, \ldots, p-1$ such that for any $m = 0, \ldots, p-1$, the set of elements $\{0, \ldots, m\}$ is an optimal ideal. Let $(P', \leq_P)$ be the subposet poset of P with the element set $\{0, 1, \ldots, p-2\}$ and the induced partial order. It is easily seen that $\{0, 1, \ldots, m\}$ is an optimal ideal in P' for $m = 0, 1, \ldots, p-2$.

Let $H'(P')$ be the hypergraph represented by $(P', \leq_P)$ by induction with $V_{H'} = \{0, \ldots, p-2\}$. Denote $V_i = \{j \in V_{H'} \mid r_P(j) \leq i\}$. The hypergraph $H(P)$ is obtained from $H'(P')$ by adding a new vertex v (with label $p-1$) and several new hyperedges involving v. If $\omega_P(p-1) = 0$ then v is an isolated vertex. Otherwise, add to $H'(P')$ hyperedges of the form $\{v\} \cup V_i$ for $i = 0, 1, \ldots, r_P(p-1)-1$. This construction being applied to the poset in Fig. 1(a) results in the hypergraph shown in Fig. 1(b). A bit more complicated example is shown in Fig. 5. The correctness of this construction is provided by the following theorem.

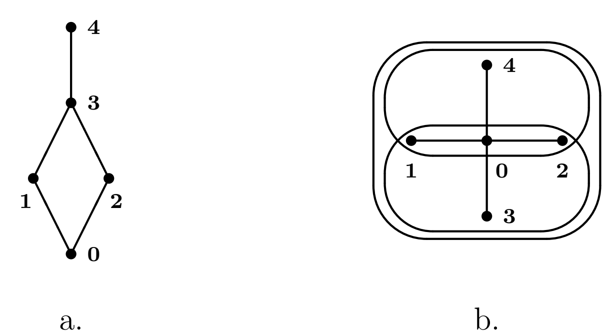


Figure 1: Construction of a hypergraph (b) represented by a given poset (a)

Theorem 1 *A ranked poset $(P, \leq_P)$ admitting an MRI-order represents the hypergraph $H = H(P)$ constructed above. Moreover, H admits an E-order.*

Proof.
We first show that the poset $(P, \leq_P)$ satisfies the following condition:

$$\omega_P(m) - \omega_P(m-1) \leq 1 \quad \text{for} \quad m = 1, \ldots, |P| - 1. \tag{5}$$

Let us denote $p = |P|$ and label the elements of P by numbers $0, 1, \ldots, p-1$ according to the MRI-order. Assume there exists an index $i \geq 1$ for which $\omega_P(i) - \omega_P(i-1) \geq 2$. Note that $\omega_P(i) = r_P(i)$ for all i. The key observation is that for the elements $x = i-1 \in P$ and $y = i \in P$ one has $x \not<_P y$ and $y \not<_P x$. Indeed, if $y <_P x$ then $r_P(y) < r_P(x)$. So, the initial segment $\{0, \ldots, x\}$ of the MRI-order is not an ideal, which is a contradiction. Similarly, if $x <_P y$ then the condition $r_P(y) \geq r_P(x) + 2$ implies the existence of an element $z \in P$ with $r_P(z) = r_P(x)+1$ and $x <_P z <_P y$. Consider the ideal $A = \{0, \ldots, y\}$. Since x and y are the two last elements in the ideal (in the MRI-order), $z < x$. But then for the ideal $B = \{0, \ldots, z\}$ one has $x \notin B$, which is again a contradiction.

Therefore, the sets $C = \{0, 1, \ldots, x\}$ and $D = \{0, 1, \ldots, i-2, y\}$ are ideals. Moreover, C is an initial segment of the MRI-order and D is not. However, $R_P(C) - R_P(D) = \omega_P(i-1) - \omega_P(i) < 0$, which contradicts the optimality of C. So, (5) is established.

To prove that H is represented by $(P, \leq_P)$ we have to show that $E_H(m) = R_P(m)$ for $m = 1, \ldots, p$. The construction of H provides $|E(\{0, 1, \ldots, i\})| - |E_H(\{0, 1, \ldots, i-1\})| = \omega_P(i)$ for $i = 1, \ldots, p-1$. Therefore,

$$\begin{aligned} E_H(m) &= |E_H(\{0, 1, \ldots, m-1\})| \\ &= \sum_{i=1}^{m-1} (|E_H(\{0, 1, \ldots, i\})| - |E_H(\{0, 1, \ldots, i-1\})|) \\ &= \sum_{i=1}^{m-1} \omega_P(i) = R_P(m). \end{aligned}$$

Finally, we prove that H admits an E-order by showing that $\{0, 1, \ldots, m-1\} \subseteq V_H$ is an optimal set for $m = 1, \ldots, p$. For this consider an optimal set $A \subseteq V_H$ with $|A| = m$. Denote $a = \max_{x \in A} x$ and $b = \min_{x \notin A} x$. Without loss of generality we can assume $b < a$. If we consider the vertices a and b as the corresponding elements of P, then $r_P(a) = \omega_P(a)$ and $r_P(b) = \omega_P(b)$. By the construction of H,

$$\begin{aligned} |E_H(A \cup \{b\})| - |E_H(A)| &\geq \omega_P(b) \\ |E_H(A)| - |E_H(A \setminus \{a\})| &\leq \omega_P(a). \end{aligned}$$

Hence, if $\omega_P(b) \geq \omega_P(a)$ then swapping a and b results in a set B with $|E_H(B)| \geq |E_H(A)|$.

Assume $\omega_P(b) < \omega_P(a)$, i.e., $r_P(b) < r_P(a)$. By the construction of H, for any $e \in E_H(A)$ such that $a \in e$ and for any $x \in e$ one has $r_P(x) < r_P(b)$ (since otherwise $b \in A$). Since b is the vertex with the smallest label which is not in A, the number of hyperedges in $E(A)$ that contain a is $r_P(b) = \omega_P(b)$. Hence, if we swap a and b, we again obtain a set $B \subseteq V_H$ with $|E_H(B)| \geq |E_H(A)|$. In either case B is an optimal set. Applying these swappings sufficiently many times, we can transform any optimal set A into the set $\{0, 1, \ldots, m-1\}$ without decreasing the number of the inner hyperedges. □

Since we will be dealing with cartesian products of hypergraphs, it would be nice to have a product theorem for the representation. To formulate this theorem we need the following definition. Let $(P', \leq_{P'})$ and $(P'', \leq_{P''})$ be posets. The cartesian product of these posets is a poset with the element set $P' \times P''$ and the partial order $\leq_\times$ defined as follows:

$(x', x'') \leq_\times (y', y'')$ iff $x' \leq_{P'} x''$ and $y' \leq_{P''} y''$. It is easily shown that if $(P', \leq_{P'})$ and $(P'', \leq_{P''})$ are ranked posets then their cartesian product is a ranked poset too, moreover $r_{P' \times P''}(x, y) = r_{P'}(x) + r_{P''}(y)$. An analog of the following result for graphs is proved in [6]. This proof essentially works for hypergraphs too, so we do not repeat it here.

Theorem 2 *Let $H_1, \ldots, H_n$ be hypergraphs that are represented by posets $P_1, \ldots, P_m$, respectively. If each $H_1, \ldots, H_n$ admits an E-order, then $H_1 \times \cdots \times H_n$ is represented by $P_1 \times \cdots \times P_n$.*

Therefore, we have reduced the EIP on a hypergraph H and its cartesian powers to the MRI problem on the representing posets and shown that both problems are equivalent. An advantage of dealing with posets is that a solution to the MRI problem can, in turn, be deduced from a solution to another extremal poset problem. Without going into details, for which the readers are referred to [6], let us mention that any *Macaulay* poset admits an MRI-order. Hence, any Macaulay poset represents a hypergraph that admits an E-order.

In the literature, there exist many results on Macaulay posets representable as cartesian powers of other posets. By Theorem 1, every such result is relevant to the EIP on some hypergraph. Some applications of this approach are presented in the next section.

3 Application of poset results to hypergraphs

Here we present solutions to the EIP for the cartesian powers of some hypergraphs. We obtain these results as corollaries of some known results on Macaulay posets, provided by the theory outlined in Section 2. For a hypergraph H introduce its *δ-sequence* $\delta(H) = (\delta_H(0), \ldots, \delta_H(|H| - 1))$.

The definitions of the MRI-orders in the following examples involve the *lexicographic order* $\leq_{lex}$ that plays a particular role in this area. The lexicographic order is defined on the set of all n-dimensional ($n \geq 1$) vectors with integral entries. For two such vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ we say that $\mathbf{x} <_{lex} \mathbf{y}$ iff $x_i = y_i$ for $i = 1, \ldots, t$ and $x_{t+1} < y_{t+1}$ for some t, $0 \leq t < n$.

3.1 The star hypergraphs

The hypergraph T_k is defined by its δ-sequence:

$$\delta(T_k) = (\underbrace{0, 0, \ldots, 0}_{k-1}, 1)$$

for $k > 1$. Therefore, T_k has k vertices and the only hyperedge including them all. The hypergraph T_6 and the Hasse diagram of its representing poset is shown in Fig. 2(a) and Fig. 2(b), respectively.

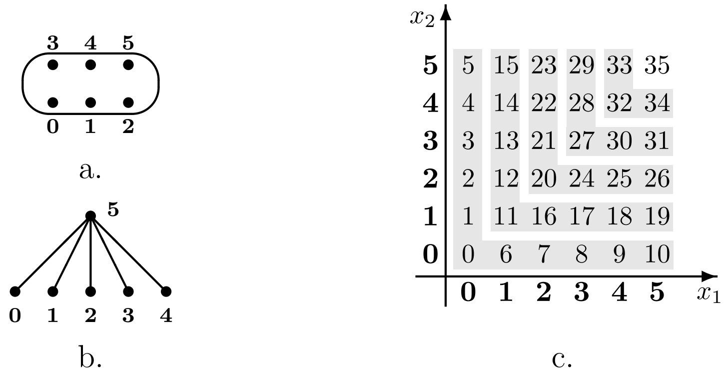


Figure 2: The star hypergraph T_6 (a), its representing poset (b), and the order $\mathcal{T}_6^2$ (c)

The MRI problem for the cartesian powers of the representing poset has been studied by several authors (see [6] for references). For the labeling of vertices of this poset shown in Fig. 2(c), it is known that the inverse of following order $\mathcal{T}_k^n$ (known in the literature as *salami order*) is an MRI-order for the n-th power of the representing poset.

For $\mathbf{x} = (x_1, \ldots, x_n) \in V_{T_k^n}$ denote $\overline{\mathbf{x}} = \max_i x_i$ and let $\tilde{\mathbf{x}}$ be the vector obtained from $\mathbf{x}$ by replacing all entries not equal to $\overline{\mathbf{x}}$ with 0. The order $\mathcal{T}_k^n$ is defined by the double induction on n and k. For $\mathbf{x}, \mathbf{y} \in V_{T_k^n}$ we say $\mathbf{x} >_{\mathcal{T}_k^n} \mathbf{y}$ iff

(*i*) $\overline{\mathbf{x}} > \overline{\mathbf{y}}$, or

(*ii*) $\overline{\mathbf{x}} = \overline{\mathbf{y}}$ and $\tilde{\mathbf{x}} >_{lex} \tilde{\mathbf{y}}$, or

(*iii*) $\overline{\mathbf{x}} = \overline{\mathbf{y}} = t > 1$, $\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$, and $\mathbf{x}' >_{\mathcal{T}_{t-1}^{n-|I|}} \mathbf{y}'$, where $\mathbf{x}', \mathbf{y}'$ are obtained from $\mathbf{x}, \mathbf{y}$ respectively by deleting all entries x_i, y_i with $i \in I = \{j \mid x_j = y_j = t\}$.

Therefore we first order lexicographically all n-dimensional vectors with binary entries, and in the binary case ($k = 1$) the order $\mathcal{T}_1^n$ is just the lexicographic order. The example below shows the vertices of $V_{T_2^3}$ in the increasing order $\mathcal{T}_2^3$ (we omitted the commas and parenthesis in all vectors):

000 001 010 011 100 101 110 111 002 012 102 112 020 021 120 121 022 122 200 201 210 211 202 212 220 221 222.

To get the inverse of this order $\tilde{\mathcal{T}}_k^n$ one has to replace in this list $(x_1, \ldots, x_n)$ with $(k - x_1, \ldots, k - x_n)$ and put the obtained vectors in the reversed order. This operation for $n = 2$ and $k = 6$ results in the order $\tilde{\mathcal{T}}_6^2$ schematically shown in Fig. 2(c). Any initial segment of the order $\tilde{\mathcal{T}}_k^n$ constitutes an optimal ideal in the n-th cartesian power of the representing poset [6], and, hence, also an optimal set in the hypergraph T_k^n.

3.2 The spider hypergraphs

This hypergraph $R_{k,t}$ for $k > 1$ and $t \geq 1$ has the following δ-sequence:

$$\delta(R_{k,t}) = (0, 1, \dots, t-1, 0, 1, \dots, t-1, \dots, 0, 1, \dots, t-1, t),$$

where the pattern $[0, 1, \dots, t-1]$ repeats k times. This hypergraph is a generalization of the star hypergraph T_k. It consists of $kt + 1$ vertices and $k\binom{t+1}{2}$ hyperedges. For $i = 0, 1, \dots, t-1$ denote $V_i = \{x \in V_{R_{k,t}} \mid (x \bmod t) \leq i\}$. Then the hyperedge set of $R_{k,t}$ is defined by

$$E_{R_{k,t}} = \left\{(x, y) \mid 0 \leq x, y < kt,\ \left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{y}{k} \right\rfloor\right\} \cup \bigcup_{i=0}^{t-1} \{\{kt+1\} \cup V_i\}.$$

Any hyperedge of the first part of the union consists of two vertices, so it is just an edge. The hyperedges of the second part consist of $k+1, 2k+1, \dots, tk+1$ vertices. An example of $R_{3,3}$ is shown in Fig. 3(a). The representing poset of $R_{3,3}$ is known as the spider poset and was studied in [7].

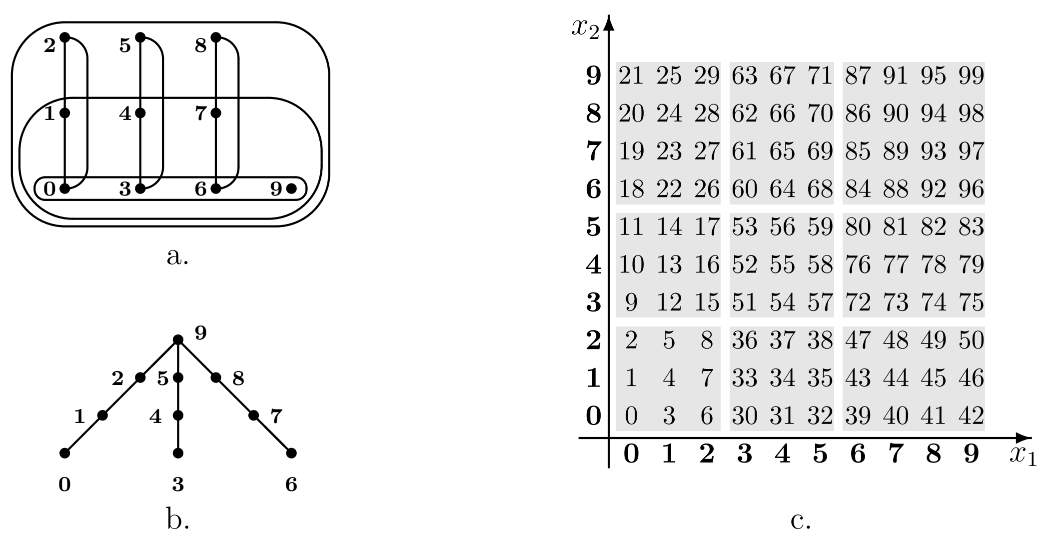


Figure 3: The hypergraph $R_{3,3}$ (a), its representing poset (b), and the order $\mathcal{R}^2_{3,3}$ (c).

To define an MRI-order on $V^n_{R_{k,t}}$, following [7], for $x \in V_{R_{k,t}}$ denote

$$\underline{x} = \begin{cases} \lfloor x/t \rfloor, & \text{if } x \neq kt \\ k-1, & \text{if } x = kt \end{cases}$$

Hence, $0 \leq \underline{x} \leq k-1$. For $\mathbf{x} = (x_1, \dots, x_n) \in V_{R^n_{k,t}}$ let $\underline{\mathbf{x}} = (\underline{x}_1, \dots, \underline{x}_n)$.

Furthermore, for $i = 0, 1, \dots, k-1$ and $\mathbf{x} = (x_1, \dots, x_n) \in V_{R^n_{k,t}}$ let $I_i(\mathbf{x}) = \{j \mid \underline{x}_j = i\}$. If $I_i(\mathbf{x}) = \{j_1, \dots, j_q\}$ for some $q \geq 1$ and $j_1 < \dots < j_q$, then denote $p_i(\mathbf{x}) = (x_{j_1}, \dots, x_{j_q})$. We define the vector $\vec{\mathbf{x}} = p_0(\mathbf{x})p_1(\mathbf{x})\dots p_{k-1}(\mathbf{x})$ as the concatenation of vectors $p_i(\mathbf{x})$,

$i = 0, \ldots, k-1$. If $I_i(\mathbf{x}) = \emptyset$ for some i, then the corresponding term $p_i(\mathbf{x})$ is not present in the concatenation. Therefore, $\vec{\mathbf{x}}$ is obtained from $\mathbf{x}$ by some permutation of its entries.

For example, if $\mathbf{x} = (0, 8, 1, 3, 5, 4) \in V_{R^6_{3,3}}$, then $\underline{\mathbf{x}} = (0, 2, 0, 1, 1, 1)$, $I_0(\mathbf{x}) = \{1, 3\}$, $I_1(\mathbf{x}) = \{4, 5, 6\}$, $I_2(\mathbf{x}) = \{2\}$ $p_0(\mathbf{x}) = (0, 1)$, $p_1(\mathbf{x}) = (3, 5, 4)$, $p_2(\mathbf{x}) = (8)$, and $\vec{\mathbf{x}} = (0, 1, 3, 5, 4, 8)$.

Now we define an auxiliary total order $\mathcal{Z}^n_k$ on the set $Z^n_k = \{(z_1, \ldots, z_n) \mid z_i \in \{0, 1, \ldots, k-1\},\ 1 \le i \le n\}$. The definition is inductive on k and n. Since $|Z^n_1| = 1$, then the order $\mathcal{Z}^n_0$ is trivial. For $n = 1$ we order the elements of Z^1_k as $0 < 1 < \cdots < k$. Assume the order $\mathcal{Z}^n_{k'}$ is defined for any k' and n' with $0 \le k' < k$ and $n' < n$, and let $k \ge 2$ and $n \ge 2$.

First, we partition the elements of Z^n_k into 2^n *blocks* $B^1, \ldots, B^{2^n}$. For this consider the set

$$\hat{Z}^n_k = \{(z_1, \ldots, z_n) \mid z_i \in \{0, k-1\},\ 1 \le i \le n\}$$

and order its elements (which we denote by $\mathbf{b}^1, \ldots, \mathbf{b}^{2^n}$) lexicographically. The block B^i consists of the vectors which can be obtained from $\mathbf{b}^i = (b^i_1, \ldots, b^i_n)$ by replacing its any non-zero entry $b^i_j = k-1$ with some element of the set $\{1, \ldots, k-1\}$, $j = 1, \ldots, n$. Thus, if $\mathbf{b}^i$ has s nonzero entries, then B^i consists of $(k-1)^s$ vectors of Z^n_k.

Let $\mathbf{x}, \mathbf{y} \in Z^n_k$, $\mathbf{x} \in B^i$ and $\mathbf{y} \in B^j$. If $i > j$, we put $\mathbf{x} > \mathbf{y}$ in order $\mathcal{Z}^n_k$. If $i = j$ then let s be the number of the common zero entries of $\mathbf{x}$ and $\mathbf{y}$. Consider the vectors $\mathbf{x}'$ and $\mathbf{y}'$ obtained from $\mathbf{x}$ and $\mathbf{y}$ respectively by omitting their zero entries (if $s > 0$, i.e. $i < 2^n$) and decreasing every remaining entry by 1. Note that $\mathbf{x}', \mathbf{y}' \in Z^{n-s}_{k-1}$. We put $\mathbf{x} \ge \mathbf{y}$ in order $\mathcal{Z}^n_k$ iff $\mathbf{x}' \ge \mathbf{y}'$ in order $\mathcal{Z}^{n-s}_{k-1}$.

As an example we present the order $\mathcal{Z}^2_4$ below. In order to simplify the notations we omitted commas and parenthesis in vectors. The generating vectors of the blocks (i.e. the vectors of $\hat{Z}^n_k$) are shown in bold. Since the order $\mathcal{Z}^2_4$ involves the orders $\mathcal{Z}^1_2$, $\mathcal{Z}^1_3$, $\mathcal{Z}^2_2$, and $\mathcal{Z}^2_3$, we present them too:

$$\mathcal{Z}^1_2 : \underbrace{\mathbf{0}}_{B_1}\ \underbrace{\mathbf{1}}_{B_2}; \qquad \mathcal{Z}^1_3 : \underbrace{\mathbf{0}}_{B_1}\ \underbrace{1\ \mathbf{2}}_{B_2}; \qquad \mathcal{Z}^2_2 : \underbrace{\mathbf{00}}_{B_1}\ \underbrace{\mathbf{01}}_{B_2}\ \underbrace{\mathbf{10}}_{B_3}\ \underbrace{\mathbf{11}}_{B_4}$$

$$\mathcal{Z}^2_3 : \underbrace{\mathbf{00}}_{B_1}\ \underbrace{01\ \mathbf{02}}_{B_2}\ \underbrace{10\ \mathbf{20}}_{B_3}\ \underbrace{11\ 12\ 21\ \mathbf{22}}_{B_4}$$

$$\mathcal{Z}^2_4 : \underbrace{\mathbf{00}}_{B_1}\ \underbrace{01\ 02\ \mathbf{03}}_{B_2}\ \underbrace{10\ 20\ \mathbf{30}}_{B_3}\ \underbrace{11\ 12\ 13\ 21\ 31\ 22\ 23\ 32\ \mathbf{33}}_{B_4}.$$

Now we are ready to define the total order $\mathcal{R}^n_{k,t}$ on $V_{R^n_{k,t}}$. For $\mathbf{x}, \mathbf{y} \in V_{R^n_{k,t}}$, $n \ge 1$ and $t \ge 1$ we write $\mathbf{y} <_{\mathcal{R}^n_{k,t}} \mathbf{x}$ iff

(*i*) $\underline{\mathbf{y}} <_{\mathcal{Z}^n_k} \underline{\mathbf{x}}$, or

(*ii*) $\underline{\mathbf{x}} = \underline{\mathbf{y}}$ and $\vec{\mathbf{x}} >_{lex} \vec{\mathbf{y}}$.

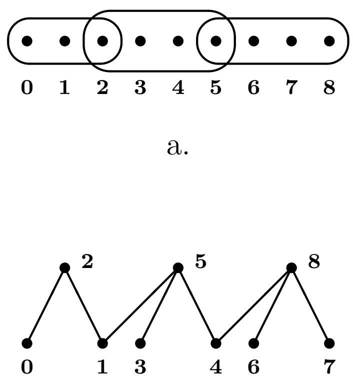
An example of the order $\mathcal{R}^2_{3,3}$ is shown in Fig. 3(c).

3.3 The chain hypergraphs

The hypergraph $P_{k,t}$ for given $k > 1$ and $t \geq 1$ has the following δ-sequence:

$$\delta(P_{k,t}) = (\underbrace{0,\ldots,0}_{k-1},1,\underbrace{0,\ldots,0}_{k-1},1,\ldots,1,\underbrace{0,\ldots,0}_{k-1},1),$$

where the pattern $[0,\ldots,0,1]$ repeats t times. This hypergraph has $(k-1)t$ vertices and t hyperedges of size $k-1,k,\ldots,k$. The hypergraph $P_{4,3}$ and its representing poset are shown in Fig. 4(a) and Fig. 4(b), respectively.



a.

b.

x_2 \ x_1	0	1	2	3	4	5	6	7	8
8	56	60	62	65	69	71	74	78	80
7	55	59	61	64	68	70	73	77	79
6	54	57	58	63	66	67	72	75	76
5	20	24	26	29	33	35	47	51	53
4	19	23	25	28	32	34	46	50	52
3	18	21	22	27	30	31	45	48	49
2	2	6	8	11	15	17	38	42	44
1	1	5	7	10	14	16	37	41	43
0	0	3	4	9	12	13	36	39	40

c.

Figure 4: The hypergraph $P_{4,3}$ (a), its representing poset (b), and the order $\mathcal{P}^2_{4,3}$ (c).

The MRI-order $\mathcal{P}^n_{k,t}$ for the cartesian powers of the chain hypergraphs follows from a general construction in [8]. For $x \in V_{P_{k,t}}$ denote $\underline{x} = x \bmod k$ and $\overline{x} = \lfloor x/k \rfloor$. We write $(x_1,\ldots,x_n) <_{\mathcal{P}^n_{k,t}} (y_1,\ldots,y_n)$ iff

(*i*) $(\underline{x}_1,\ldots,\underline{x}_n) <_{lex} (\underline{y}_1,\ldots,\underline{y}_n)$, or

(*ii*) $(\underline{x}_1,\ldots,\underline{x}_n) = (\underline{y}_1,\ldots,\underline{y}_n)$ and $(\overline{x}_1,\ldots,\overline{x}_n) <_{\mathcal{T}^n_k} (\overline{y}_1,\ldots,\overline{y}_n)$,

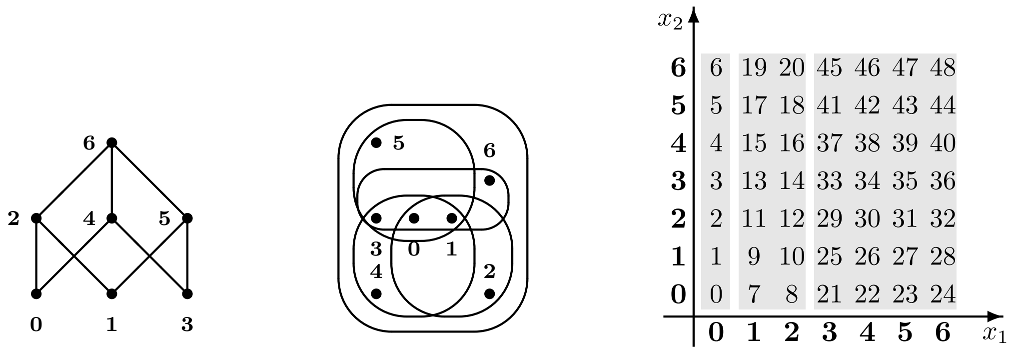
where the order $\mathcal{T}^n_k$ is defined above. An example of the order $\mathcal{P}^2_{4,3}$ is shown in Fig. 4(c).

3.4 The simplicial hypergraphs

The simplicial hypergraph S_d is defined by its representing poset, which is the Boolean lattice of dimension $d > 1$ (or d-cube) without the vertex of rank 0 (cf. Fig. 5(a) for $d = 3$). Therefore, the δ-sequence of S_d can be obtained from the δ-sequence of Q^d by deleting its first entry and decreasing all the other entries by 1. The hypergraph itself is constructed according to the procedure described in Section 2. For example, for $d = 3$ the δ-sequence of Q^d is $\delta(Q^3) = (0,1,1,2,1,2,2,3)$, so

$$\delta(S_3) = (0,0,1,0,1,1,2).$$

Thus, S_d has $2^d - 1$ vertices and $\sum_{k=2}^{d} \binom{d}{k}(k-1)$ edges. We view the vertices of S_d as binary strings of length d, similarly as the corresponding vertices of Q^d, and label the vertex corresponding to the string $(x_1, \ldots, x_d)$ by $\sum_{i=1}^{d} x_i 2^i - 1$. The representing poset of S_3 and the hypergraph itself are shown in Fig. 5(a) and 5(b), respectively.



a. b. c.

Figure 5: The representing poset of S_3 (a), the hypergraph (b), and the order $\mathcal{S}_3^2$ (c).

The MRI-order $\mathcal{S}_d^n$ for the cartesian product of n simplicial hypergraphs S_d can be derived from the recent result concerning the submatrix order [17]. The order $\mathcal{S}_d^n$ is particularly simple for $n = 2$. To describe it we first partition the vertices of S_d into d blocks B_i, where $B_i = \{z \in V_{S_d} \mid 2^{i-1} - 1 \leq z < 2^i - 1\}$ for $i = 1, 2, \ldots, d$ and denote by $B(z)$ the number of the block containing z. Now, for $(x_1, x_2),\ (y_1, y_2) \in V_{S_d^2}$ we write $(x_1, x_2) <_{\mathcal{S}_d^2} (y_1, y_2)$ iff

(*i*) $B(x_1) < B(y_1)$, or

(*ii*) $B(x_1) = B(y_1)$ and $(x_2, x_1) <_{lex} (y_2, y_1)$.

The order $\mathcal{S}_d^n$ for $n > 2$ can be derived from [17]. An example of the order $\mathcal{S}_3^2$ is shown in Fig. 5(c), where the blocks are shadowed.

4 Compression

In this section we extend to hypergraphs two known results for graphs based on the compression [5]. These results will be used in the next two sections. For $n \geq 2$, $\mathbf{x} = (x_1, \ldots, x_{n-1}) \in V_{H^{n-1}}$, $x \in V_H$, and $A \subseteq V_{H^n}$ denote

$$\begin{aligned} A_i(\mathbf{x}) &= \{z \in V_H \mid (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \in A\}, \\ A_{\bar{i}}(x) &= \{(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \in V_{H^{n-1}} \mid (z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \in A\}. \end{aligned}$$

Suppose we got a series of total orders $\{\mathcal{O}^k\}_{k \geq 1}$ with $\mathcal{O}^k$ being defined on V_{H^k}. For $1 \leq m \leq |V_{H^n}|$, denote by $\mathcal{F}_{\mathcal{O}^n}(m)$ the *initial segment* of the order $\mathcal{O}^n$ of length m.

We say that the set A is *weakly i-compressed* if $A_i(\mathbf{x}) = \mathcal{F}_{\mathcal{O}^1}(|A_i(\mathbf{x})|)$ for any $\mathbf{x} \in V_{H^{n-1}}$. We call A *weakly compressed* if it is weakly i-compressed for $i = 1, 2, \ldots, n$. Similarly, we say that A is *i-compressed* if $A_{\bar{i}(x)} = \mathcal{F}_{\mathcal{O}^{n-1}}(|A_{\bar{i}}(x)|)$ for any $x \in V_H$. We call A *compressed* if it is i-compressed for $i = 1, 2, \ldots, n$.

Assume $\mathbf{x} = (x_1, \ldots, x_n) \in V_{H_d^n}$ and $\mathbf{y} = (y_1, \ldots, y_n) \in V_{H_d^n}$ with $x_i = y_i$. Denote by $\mathbf{x}', \mathbf{y}'$ the vectors obtained, respectively, from $\mathbf{x}$ and $\mathbf{y}$ by deleting their i-th entries. For $n \geq 2$ we call the order $\mathcal{O}^n$ *consistent* if for any $i = 1, \ldots, n$, and any $\mathbf{x}, \mathbf{y}$ with $x_i = y_i$ as above, one has $\mathbf{x}' <_{\mathcal{O}^{n-1}} \mathbf{y}'$ iff $\mathbf{x} <_{\mathcal{O}^n} \mathbf{y}$.

Lemma 2 *Let H be a hypergraph and $A \subseteq V_{H^n}$ for some $n \geq 2$.*

(a) If H admits an E-order, then there exists a weakly compressed set $B \subseteq V_{H^n}$ such that $|B| = |A|$ and $|E_{H^n}(B)| \geq |E_{H^n}(A)|$.

(b) Let $\{\mathcal{O}^k\}_{k\geq 1}$ be a series of consistent total orders such that $\mathcal{O}^{n-1}$ is an E-order on H^{n-1}. Then there exists a compressed set $B \subseteq V_{H^n}$ such that $|B| = |A|$ and $|E_{H^n}(B)| \geq |E_{H^n}(A)|$.

Proof.
To prove the first statement, represent $H^n = H \times H^{n-1}$. For a fixed i, $1 \leq i \leq n$, one has

$$\begin{aligned} |E_{H^n}(A)| &= \sum_{\mathbf{x} \in V_{H^{n-1}}} |E_H(A_i(\mathbf{x}))| + |E_\times(A)|, \quad \text{where} \qquad (6) \\ E_\times(A) &= \{e \in E_{H^n}(A) \mid e \not\subseteq A_i(\mathbf{x}) \text{ for any } \mathbf{x} \in V_{H^{n-1}}\}. \end{aligned}$$

It is easily seen that

$$|E_\times(A)| = \sum_{e \in E_{H^{n-1}}} \left| \bigcap_{\mathbf{x} \in e} A_i(\mathbf{x}) \right| \leq \sum_{e \in E_{H^{n-1}}} \min_{\mathbf{x} \in e} |A_i(\mathbf{x})|. \qquad (7)$$

Hence, (6) and (7) imply the following upper bound

$$\begin{aligned} |E_{H^n}(A)| &= \sum_{\mathbf{x} \in V_{H^{n-1}}} |E_H(A_i(\mathbf{x}))| + |E_\times(A)| \\ &\leq \sum_{\mathbf{x} \in V_{H^{n-1}}} |E_H(A_i(\mathbf{x}))| + \sum_{e \in E_{H^{n-1}}} \min_{\mathbf{x} \in e} |A_i(\mathbf{x})|. \qquad (8) \end{aligned}$$

For a fixed i and the weakly i-compressed set $C_i(A)$ that is obtained from A by replacing $A_i(\mathbf{x})$ with $\mathcal{F}_{\mathcal{O}^1}(|A_i(\mathbf{x})|)$ for every $\mathbf{x} \in V_{H^{n-1}}$, the upper bound (7) is tight. That is,

$$|E_{H^n}(C_i(A))| = \sum_{\mathbf{x} \in V_{H^{n-1}}} |E_H(C_i(A_i(\mathbf{x})))| + \sum_{e \in E_{H^{n-1}}} \min_{\mathbf{x} \in e} |C_i(A_i(\mathbf{x}))|. \qquad (9)$$

Since H admits an E-order and $|C_i(A_i(\mathbf{x}))| = |A_i(\mathbf{x})|$ for any $\mathbf{x} \in H^{n-1}$, one gets $|E_H(C_i(A_i(\mathbf{x})))| \geq |E_H(A_i(\mathbf{x}))|$. So, (8) and (9) imply

$$\begin{aligned} |E_{H^n}(C_i(A))| &= \sum_{\mathbf{x} \in V_{H^{n-1}}} |E_H(C_i(A_i(\mathbf{x})))| + \sum_{e \in E_{H^{n-1}}} \min_{\mathbf{x} \in e} |C_i(A_i(\mathbf{x}))| \\ &\geq \sum_{\mathbf{x} \in V_{H^{n-1}}} |E_H(A_i(\mathbf{x}))| + \sum_{e \in E_{H^{n-1}}} \min_{\mathbf{x} \in e} |A_i(\mathbf{x})| = |E_{H^n}(A)|. \end{aligned}$$

Denote $S(A) = \sum_{(x_1,\ldots,x_n)\in A}\sum_{i=1}^n x_i$. It is easily seen that $S(C_i(A)) \le S(A)$ where a strict inequality holds iff $C_i(A) \neq A$. Since $S(A) \ge 0$ for any $A \subseteq V_{H^n}$, applying the above compression operation for $i = 1, 2, \ldots, n$ sufficiently many times, we will obtain a weakly compressed set B satisfying the first statement. The second statement of the lemma can be proved similarly. □

For a hypergraph H admitting an E-order, we will always assume that its vertices are labeled by $0, 1, \ldots, |V_H| - 1$ according to the E-order. For a weakly compressed set $A \subseteq V_{H^n}$ the number $|E_{H^n}(A)|$ can be computed as follows:

Lemma 3 *Let H be a hypergraph and let $A \subseteq V_{H^n}$ for some $n \ge 1$ be a weakly compressed set. Then*

$$|E_{H^n}(A)| = \sum_{(x_1,\ldots,x_n)\in A}\sum_{i=1}^n \delta_H(x_i).$$

Proof.
We prove the statement by the double induction on $|A|$ and n. It is obviously true for $n = 1$. It is also true for $|A| = 1$ and any n, since the only weakly compressed set of that size is the set $\{(0,\ldots,0)\}$. For $n \ge 2$ choose $\mathbf{y} = (y_1,\ldots,y_n) \in A$ such that $(y_1,\ldots,y_{i-1},y_i+1,y_{i+1},\ldots,y_n) \notin A$ for $i = 1,\ldots,n$. Since A is finite, the vertex $\mathbf{y}$ does exist. Since A is weakly compressed, one has

$$V_{H^n} \cap \bigcup_{i=1}^n \{(y_1,\ldots,y_{i-1},y_i-1,y_{i+1},\ldots,y_n)\} \subseteq A.$$

Denote $B = A \setminus \{\mathbf{y}\}$. Then B is weakly compressed, $B_i(\mathbf{y}) \subseteq A_i(\mathbf{y})$, and $A_i(\mathbf{y})$ and $B_i(\mathbf{y})$ are weakly compressed. Using the induction hypothesis, one has

$$\begin{aligned}
|E_{H^n}(A)| &= |\{e \in E_{H^n}(A) \mid \mathbf{y} \notin e\}| + |\{e \in E_{H^n}(A) \mid \mathbf{y} \in e\}| \\
&= |E_{H^n}(B)| + \left|\bigcup_{i=1}^n E_H(A_i(\mathbf{y}) \setminus B_i(\mathbf{y}))\right| \\
&= |E_{H^n}(B)| + \sum_{i=1}^n (|E_H(A_i(\mathbf{y}))| - |E_H(B_i(\mathbf{y}))|) \\
&= |E_{H^n}(B)| + \sum_{i=1}^n \left(\sum_{\mathbf{x}\in A_i(\mathbf{y})}\sum_{j=0}^{y_i}\delta_H(j) - \sum_{\mathbf{x}\in B_i(\mathbf{y})}\sum_{j=0}^{y_i-1}\delta_H(j)\right) \\
&= \sum_{(x_1,\ldots,x_n)\in B}\sum_{i=1}^n \delta_H(x_i) + \sum_{i=1}^n \delta_H(y_i) = \sum_{(x_1,\ldots,x_n)\in A}\sum_{i=1}^n \delta_H(x_i),
\end{aligned}$$

which completes the proof. □

It can be shown for a series $\{\mathcal{O}^k\}_{k\ge1}$ of consistent total orders that if $A \subseteq V_{H^n}$ is a compressed set, then A is also weakly compressed (counterexamples show that the inverse of this is not always true). Therefore, Lemma 3 also holds for compressed sets.

5 A purely hypergraph result

In this section we present a solution to the EIP problem for a new family of hypergraphs. For this family we cannot apply the poset technique developed in Section 2 because the corresponding hypergraphs do not satisfy (4).

Consider the hypergraph H_d on the vertex set of the d-cube Q^d. The hyperedges of H_d are formed by the usual edges of Q^d and by one more hyperedge including all vertices of H_d. Hence, H_d has 2^d vertices and $d2^{d-1}+1$ edges. The δ-sequence of H_d is obtained from the one of Q^d by incrementing its last entry on 1. For example,

$$\delta(Q^2) = (0,1,1,2) \qquad \delta(H_2) = (0,1,1,3)$$
$$\delta(Q^3) = (0,1,1,2,1,2,2,3) \qquad \delta(H_3) = (0,1,1,2,1,2,2,4).$$

We define a total order $\mathcal{H}_d^n$ on the n-th cartesian power H_d^n of H_d as follows. For a vertex $\mathbf{x} = (x_1,\ldots,x_n) \in V_{H_d^n}$ denote by $\text{ind}(\mathbf{x})$ the minimum index i such that $x_i \leq 2^{d-1}-1$. If $x_i \geq 2^{d-1}$ for $i = 1,\ldots,n$ we set $\text{ind}(\mathbf{x}) = \infty$. The definition of the order $\mathcal{H}_d^n$ is done by the double induction on d and n. For $n = 1$ or $d = 1$ the order $\mathcal{H}_d^n$ is just the lexicographic order on the vertex set of Q^{nd}. For $n > 1$, $d > 1$, $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ we write $\mathbf{y} <_{\mathcal{H}_d^n} \mathbf{x}$ iff

(i) $\text{ind}(\mathbf{y}) < \text{ind}(\mathbf{x})$, or

(ii) $\text{ind}(\mathbf{y}) = \text{ind}(\mathbf{x}) = i < \infty$ and $y_i < x_i$, or

(iii) $\text{ind}(\mathbf{y}) = \text{ind}(\mathbf{x}) = i < \infty$, $y_i = x_i$, and $\mathbf{y}' <_{\mathcal{H}_d^{n-1}} \mathbf{x}'$, where $\mathbf{x}', \mathbf{y}'$ are obtained respectively from $\mathbf{x}, \mathbf{y}$ by omitting their i-th entries, or

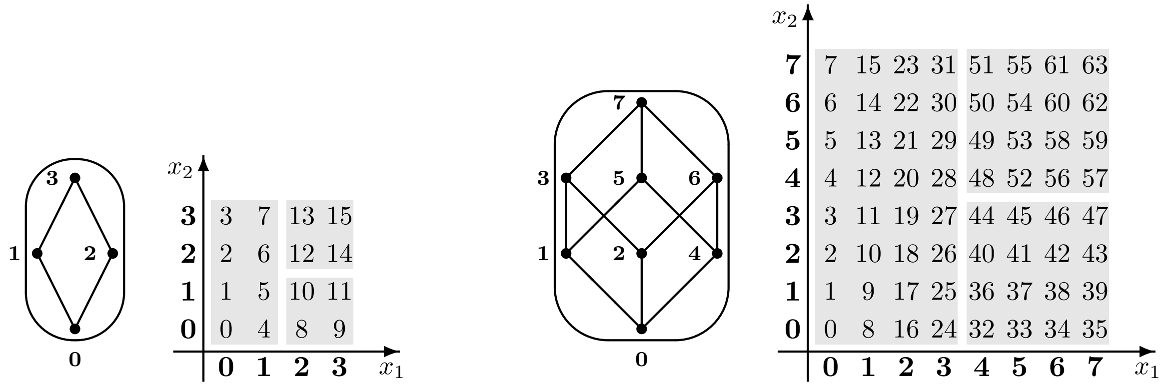
(iv) $\text{ind}(\mathbf{y}) = \text{ind}(\mathbf{x}) = \infty$ and $\hat{\mathbf{y}} <_{\mathcal{H}_{d-1}^n} \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in V_{\mathcal{H}_{d-1}^n}$ are obtained respectively from $\mathbf{x}, \mathbf{y}$ by decreasing each entry by 2^{d-1}.

The hypergraphs H_2, H_3 and the orders $\mathcal{H}_2^2, \mathcal{H}_3^2$ are shown in Fig. 6. The shadowed boxes correspond to the values of $\text{ind}(\cdot) = 1, 2$, and ∞. It is easily shown that the order $\mathcal{H}_d^n$ is consistent for $n \geq 2$.

Theorem 3 *The order $\mathcal{H}_2^n$ is an E-order on H_2^n for any $n \geq 2$.*

Proof.
Let $A \subseteq V_{H_2^n}$ be an optimal set. Denote by $\mathbf{a} = (a_1,\ldots,a_n)$ the largest element of A in order $\mathcal{H}_2^n$ and by $\mathbf{b} = (b_1,\ldots,b_n)$ the smallest element of $V_{H_d^n} \setminus A$. If A is not an initial segment of order $\mathcal{H}_2^n$ then $\mathbf{b} <_{\mathcal{H}_d^n} \mathbf{a}$.

Due to Lemma 2(b) we can assume that A is compressed. Furthermore, since the order $\mathcal{H}_d^n$ is consistent, we can assume $a_i \neq b_i$, because otherwise $\mathbf{b} \in A$. We prove the theorem by induction on n. For $n \leq 2$ it can be easily verified by using Fig. 6(a,b). We assume in the sequel that $n \geq 3$ and split the proof in 4 cases depending on the value of b_1.



a. b. c. d.

Figure 6: Hypergraphs H_2, H_3 (a,c) and the orders $\mathcal{H}_2^2, \mathcal{H}_3^2$ (b,d).

Case 1. Assume $b_1 = 0$. If $a_1 \geq 2$ then

$$\mathbf{a} = (a_1, \ldots, a_n) >_{\mathcal{H}_2^n} (b_1 + 1, b_2, \ldots, b_{n-1}, a_n) >_{\mathcal{H}_2^n} (b_1, \ldots, b_n) = \mathbf{b}.$$

Every two consecutive vectors in the above chain of inequalities have an equal entry. Since A is compressed and the order $\mathcal{H}_d^n$ is consistent, the condition $\mathbf{a} \in A$ implies $(b_1 + 1, b_2, \ldots, b_{n-1}, a_n) \in A$. By a similar reason, $\mathbf{b} \in A$, which contradicts our assumption. Hence, we can assume $a_1 = 1$. A similar argument will be used below to show $\mathbf{b} \in A$.

Claim 1 *$a_i = 0$ and $b_i > 0$ for $i = 2, \ldots, n-1$.*

Assume the contrary and let i, $2 \leq i \leq n-1$, be the smallest index for which $a_i > 0$. One has

$$\mathbf{a} = (a_1, \ldots, a_n) >_{\mathcal{H}_2^n} (a_1, \ldots, a_{i-1}, 0, \ldots, 0, b_n) >_{\mathcal{H}_2^n} (b_1, \ldots, b_n) = \mathbf{b},$$

which implies $\mathbf{b} \in A$. Now, if $a_i = 0$ for $2 \leq i \leq n-1$ then $b_i = 0$ for some i in this range also leads to a contradiction $\mathbf{b} \in A$. $\square$

For $\mathbf{x} = (x_1, \ldots, x_n)$ denote $\epsilon(\mathbf{x}) = \sum_{i=1}^n \delta_{H_d}(x_i)$. Using this denotation and Lemma 3, one has $|E_{H_d^n}(A)| = \sum_{\mathbf{x} \in A} \epsilon(\mathbf{x})$.

Claim 2 $\epsilon(\mathbf{a}) \leq \epsilon(\mathbf{b})$.

To show this, note that $a_n < b_n$, because otherwise

$$\mathbf{a} = (a_1, 0, \ldots, 0, a_n) >_{\mathcal{H}_2^n} (a_1, 0, \ldots, 0, b_n) >_{\mathcal{H}_2^n} (b_1, \ldots, b_n) = \mathbf{b}$$

and we again get $\mathbf{b} \in A$. Since $\delta(H_2) = (0, 1, 1, 3)$, the condition $a_n < b_n$ implies $\delta_{H_2}(a_n) \leq \delta_{H_2}(b_n)$. One has

$$\epsilon(\mathbf{a}) = \delta_{H_2}(a_1) + \delta_{H_2}(a_n) = 1 + \delta_{H_2}(a_n) \leq \delta_{H_2}(b_1) + \cdots + \delta_{H_2}(b_n) = \epsilon(\mathbf{b})$$

since $b_i > 0$ (Claim 1) implies $\delta_{H_2}(b_i) > 0$ for $i = 2, \ldots, n-1$. □

Note that swapping $\mathbf{a}$ and $\mathbf{b}$ results in a compressed set B. By Claim 2 and the remark preceeding it, $|E_{H_2^n}(A)| \le |E_{H_2^n}(B)|$, so B is an optimal set.

Case 2. Assume $b_1 = 1$. Then $a_1 \in \{2, 3\}$ and Claim 1 and Claim 2 can be proved in this case too. This implies that we can swap $\mathbf{a}$ and $\mathbf{b}$ to get an optimal compressed set.

Case 3. Assume $b_1 = 2$. Then $a_1 = 3$. If $(a_2, \ldots, a_n) >_{\mathcal{H}_2^{n-1}} (b_2, \ldots, b_n)$, we have

$$\mathbf{a} = (3, a_2, \ldots, a_n) >_{\mathcal{H}_2^n} (3, b_2, \ldots, b_n) >_{\mathcal{H}_2^n} (2, b_2, \ldots, b_n) = \mathbf{b},$$

which implies $\mathbf{b} \in A$. If $(a_2, \ldots, a_n) <_{\mathcal{H}_2^{n-1}} (b_2, \ldots, b_n)$ then $\mathrm{ind}(\mathbf{a}) = \mathrm{ind}(\mathbf{b}) = \infty$. Indeed, in this case $\mathrm{ind}(\mathbf{a}) = \mathrm{ind}((a_2, \ldots, a_n)) \le \mathrm{ind}((b_2, \ldots, b_n)) = \mathrm{ind}(\mathbf{b})$. On the other hand, $\mathbf{a} >_{\mathcal{H}_2^n} \mathbf{b}$ implies $\mathrm{ind}(\mathbf{a}) \ge \mathrm{ind}(\mathbf{b})$. So, $\mathrm{ind}(\mathbf{a}) = \mathrm{ind}(\mathbf{b})$. Now, if $\mathrm{ind}(\mathbf{a}) < \infty$ then (see case *(ii)* of the definition of the order) the condition $(a_2, \ldots, a_n) <_{\mathcal{H}_2^{n-1}} (b_2, \ldots, b_n)$ implies $\mathbf{a} <_{\mathcal{H}_2^n} \mathbf{b}$, which is a contradiction.

Furthermore, the first entries of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are equal to 1 and 0, respectively (the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are defined in case *(iv)* of the definition of the order $\mathcal{H}_d^n$). Hence, the argument of case 1 can be applied to $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. By this argument, if $\mathbf{b} \notin A$ then swapping $\mathbf{a}$ and $\mathbf{b}$ leads to an optimal compressed set.

Case 4. Assume $b_1 = 3$. Then $a_1 = 2$ and for $i = \mathrm{ind}(\mathbf{b})$ one has $i < \infty$, because otherwise $\mathbf{a} <_{\mathcal{H}_d^n} \mathbf{b}$. This implies $(a_2, \ldots, a_n) >_{\mathcal{H}_2^{n-1}} (b_2, \ldots, b_n)$. We show that $a_j < b_j$ for $j \notin \{1, i\}$. Indeed, if $a_j > b_j$ for some $j \notin \{1, i\}$ then

$$\mathbf{a} = (2, a_2, \ldots, a_j, \ldots, a_n) >_{\mathcal{H}_2^n} (2, a_2, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_n) >_{\mathcal{H}_2^n} (3, b_2 \ldots, b_n) = \mathbf{b}$$

which implies $\mathbf{b} \in A$, a contradiction. Note that $a_j < b_j$ implies $\delta_{H_d}(a_j) \le \delta_{H_d}(b_j)$ for $j \notin \{1, i\}$. This and $\delta_{H_d}(a_1) = \delta_{H_d}(b_1) - 2$ implies $\epsilon(\mathbf{a}) \le \epsilon(\mathbf{b})$ if $a_i \ne 3$ and $b_i \ne 0$. In this case we can swap $\mathbf{a}$ and $\mathbf{b}$ as above to get an optimal compressed set.

If $a_i = 3$ and $b_i = 0$ then the only case when $\epsilon(\mathbf{a}) > \epsilon(\mathbf{b})$ is

$$\mathbf{a} = (2, 1, \ldots, 1, 3, 1, \ldots, 1), \qquad \mathbf{b} = (3, 2, \ldots, 2, 0, 2, \ldots, 2).$$

In this case, however, $\mathbf{a} <_{\mathcal{H}_2^n} \mathbf{b}$ unless $i = 2$. But then

$$\mathbf{a} = (2, 3, 1, 1, \ldots, 1) >_{\mathcal{H}_2^n} (2, 1, 2, 1, \ldots, 1) >_{\mathcal{H}_2^n} (3, 0, 2, \ldots, 2) = \mathbf{b},$$

which implies $\mathbf{b} \in A$.

Therefore, in all cases we either show that $\mathbf{b} \in A$ or swap $\mathbf{a}$ and $\mathbf{b}$ and get an optimal compressed set. After a finite number of such swaps the starting optimal set A will be transformed into an initial segment of the order $\mathcal{H}_2^n$. □

By using a similar technique and considerably more cases, one can prove an analog of Theorem 3 for $d = 3$. We conjecture that this theorem is true for any $d \ge 2$.

6 Minimization of the hyperedge-cut

As it is mentioned in the introductory Section 1, there is no analog of identity (3) even for highly regular hypergraphs. However, in this section we present two special types of hypergraphs, for which both versions of the EIP are equivalent. The hypergraphs of the first type are those that are constructed from posets as it is described in Section 2.

Lemma 4 *Let $(P, \leq_P)$ be a poset admitting an MRI-order $\mathcal{O}$ and let $H = H(P)$ be the represented hypergraph. Then for any $A \subseteq V_H$ one has $|\theta_H(A)| \geq |\theta_H(\mathcal{F}_{\mathcal{O}}(|A|))|$.*

Proof.
By Theorem 1, H admits an E-order which we also denote by $\mathcal{O}$. For $A \subseteq H$ and $x \in V_H$ denote $\tilde{\theta}_H(x, A) = \{e \in \theta_H(A) \mid z \leq_{\mathcal{O}} x \text{ for any } z \in e\}$. In these terms,

$$|\theta_H(A)| = \sum_{x \in V_H} |\tilde{\theta}_H(x, A)|. \tag{10}$$

We can consider A as a subset of P. If $\tilde{\theta}_H(x, A) \neq \emptyset$ then

$$|\tilde{\theta}_H(x, A)| = \begin{cases} r_P(x) - \min\limits_{\substack{z \leq_{\mathcal{O}} x \\ z \notin A}} r_P(z), & \text{if } x \in A \\ r_P(x), & \text{if } x \notin A \end{cases} \tag{11}$$

Indeed, by the construction of H, one has

$$\{e \in E_H \mid z \leq_{\mathcal{O}} x \text{ for any } z \in e\} = \bigcup_{i=0}^{r_P(x)-1} \{\{x\} \cup V_i\}, \quad \text{where} \tag{12}$$
$$V_i = \{z \in P \mid z \leq_{\mathcal{O}} x \text{ and } r_P(z) \leq i\},$$

and the big union in (12) is disjoint. Hence, if $x \notin A$ then every edge under the big union in (12) is in $\theta_H(A)$. Furthermore, if $x \in A$ and $\tilde{\theta}_H(x, A) \neq \emptyset$ then there exists an element $z \in P$ such that $z \leq_{\mathcal{O}} x$ and $z \notin A$. Denote by $\underline{x}$ such an element of minimum rank. Obviously, $r_P(\underline{x}) < r_P(x)$. One has $\{x\} \cup V_i \in E_H(A)$ for $i = 0, \ldots, r_P(\underline{x}) - 1$. Therefore,

$$\tilde{\theta}_H(x, A) = \bigcup_{i=r_P(\underline{x})}^{r_P(x)-1} \{\{x\} \cup V_i\},$$

since every edge of H for i in the range above contains $\underline{x}$. This implies (11).

Denote by a the largest element of A (in the order $\mathcal{O}$) and by b the smallest element of $P \setminus A$ in this order. We show that swapping a with b leads to the set $B \subseteq V_H$ with $|\theta_H(B)| \leq |\theta_H(A)|$. For this note, that the removal of a from A only affects those edges in $\theta_H(A)$ that belong to $\tilde{\theta}_H(a, A)$. Using (10),

$$|\theta_H(B)| \leq |\theta_H(A)| + r_P(\underline{a}) - r_P(b).$$

By the choice of b one has $b \leq_{\mathcal{O}} \underline{a} <_{\mathcal{O}} a$. Hence, $r_P(b) \geq r_P(\underline{a})$ by the definition of $\underline{a}$, which completes the proof. □

Therefore, for every hypergraph constructed from a poset, the E-order is at the same time a θ-order. Our objective is to show that this also holds for the cartesian powers of those hypergraphs. The following lemma can be proved along the lines of the proof of Lemma 2.

Lemma 5 *Let H be a hypergraph and $A \subseteq V_{H^n}$ for some $n \geq 2$.*

(a) If H admits a θ-order, then there exists a weakly compressed set $B \subseteq V_{H^n}$ such that $|B| = |A|$ and $|\theta_{H^n}(B)| \leq |\theta_{H^n}(A)|$.

(b) Let $\{\mathcal{O}^k\}_{k\geq 1}$ be a series of consistent total orders such that $\mathcal{O}^{n-1}$ is an θ-order on H^{n-1}. Then there exists a compressed set $B \subseteq V_{H^n}$ such that $|B| = |A|$ and $|\theta_{H^n}(B)| \leq |\theta_{H^n}(A)|$.

We do not use Lemma 5(b) here and present it just for the completeness. The next result establishes an equivalence between the two versions of the EIP for the hypergraphs constructed from posets.

Theorem 4 *Let $(P, \leq_P)$ be a poset admitting an MRI-order and let $H = H(P)$ be the represented hypergraph. Then for any $n \geq 1$ and any m, $1 \leq m \leq |V_{H^n}|$, one has*

$$|\theta_{H^n}(m)| + |E_{H^n}(m)| = |E_{H^n}|. \tag{13}$$

Proof.
Let $\mathcal{O}$ be an MRI-order on P and $A \subseteq V_H$ be a θ-optimal set. We can consider $\mathcal{O}$ as a total order on V_H and A as a subset of P. By Lemma 4, for $n = 1$ we can assume $A = \mathcal{F}_{\mathcal{O}}(|A|)$, so $|\theta_H(A)| = \theta_H(m)$ and $|E_H(A)| = E_H(m)$. By the construction of H, one has $\tilde{\theta}_H(x, A) \neq \emptyset$ if and only if $x \notin A$. Moreover, $r_P(x) = \delta_H(x)$ for any x. Using (10) and (11), we get

$$\begin{aligned} |\theta_H(A)| &= \sum_{x\in V_H} |\tilde{\theta}_H(x, A)| = \sum_{x \notin A} r_P(x) \\ &= \sum_{x\in P} r_P(x) - \sum_{x\in A} r_P(x) \\ &= |E_H| - |E_H(A)|. \end{aligned}$$

For $n \geq 2$, the proof is based on a similar idea. Let $A \subseteq V_{H^n}$ be a weakly compressed set. Note that $r_{P^n}((x_1, \ldots, x_n)) = \sum_{i=1}^n r_P(x_i)$. Theorem 2 implies $r_{P^n}(\mathbf{x}) = \delta_{H^n}(\mathbf{x})$ for any $\mathbf{x}$. One has

$$\begin{aligned} |\theta_{H^n}(A)| &= \sum_{i=1}^{n} \sum_{\mathbf{x}\in H^{n-1}} |\theta_H(A_i(\mathbf{x}))| \\ &= \sum_{i=1}^{n} \sum_{\mathbf{x}\in H^{n-1}} \left(\sum_{y\in P} r_P(y) - \sum_{y\in A_i(\mathbf{x})} r_P(y) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(\sum_{\mathbf{x}\in P^n} r_P(x_i) - \sum_{\mathbf{x}\in A} r_P(x_i)\right) \\
&= \sum_{\mathbf{x}\in P^n}\sum_{i=1}^{n} r_P(x_i) - \sum_{\mathbf{x}\in A}\sum_{i=1}^{n} r_P(x_i) \\
&= \sum_{\mathbf{x}\in P^n} r_{P^n}(\mathbf{x}) - \sum_{\mathbf{x}\in A} r_{P^n}(\mathbf{x}) \\
&= |E_{H^n}| - |E_{H^n}(A)|.
\end{aligned}
$$

Hence, $|\theta_{H^n}(A)| + |E_{H^n}(A)| = |E_{H^n}|$ for any weakly compressed set. If $A \subseteq V_{H^n}$ is a θ-optimal set, due to Lemma 5(a) we can assume that A is weakly compressed. Then the above identity implies $|E_{H^n}(A)| = E_{H^n}(m)$, which completes the proof. □

The hypergraphs of the second type, that we consider here with respect to the minimization of the cut, are obtained from the ordinary graphs by adding to them a single hyperedge containing all the graph vertices. The hypergraph H_d considered in Section 5 belongs to this class.

If we turn a regular graph G into a hypergraph H as described above, then for any $A \subset V_H$ one has $|\theta_H(A)| = |\theta_G(A)| + 1$. Hence, A is θ-optimal in G iff it is θ-optimal in H. On the other hand, by (3), A is θ-optimal in G iff it is E-optimal (in G). This implies that both versions of the EIP are equivalent for H.

The above simple observation, however, does not hold, in general, for the products of the considered hypergraphs. Counterexamples show that the cut minimization problem for the hypergraph H_d^2, considered in the last section, has no nested solutions. However, we present below 3 examples of hypergraphs of the second type, whose any cartesian power admits a θ-order.

Denote $\tau_H(m) = \theta_H(m+1) - \theta_H(m)$ for $m = 0, \ldots, |V_H| - 1$. The following lemma can be proved similarly as Lemma 3.

Lemma 6 *Let H be a hypergraph and let $A \subseteq V_{H^n}$ for some $n \geq 1$ be a weakly compressed set. Then*

$$|\theta_{H^n}(A)| = \sum_{(x_1,\ldots,x_n)\in A} \sum_{i=1}^{n} \tau_H(x_i).$$

For $k \geq 3$, denote by $\hat{C}_k$ and $\hat{P}_k$ the hypergraphs obtained from a cycle C_k and a path P_k on k vertices, respectively, by adding to them a hyperedge containing all its vertices.

Theorem 5 *For $k = 3, 4, 5$ and any $n \geq 1$ the hypergraphs $\hat{C}_k^n$ and $\hat{P}_k^n$ admit θ-orders.*

Proof.
It is easily shown that

$$\theta_{C_k}(m) = \begin{cases} 2, & \text{for } m < k \\ 0, & \text{for } m = k \end{cases} \quad \text{and} \quad \theta_{P_k}(m) = \begin{cases} 1, & \text{for } m < k \\ 0, & \text{for } m = k \end{cases}$$

Furthermore, $\theta_{\hat{C}_k}(m) = \theta_{C_k}(m) + 1$ and $\theta_{\hat{P}_k}(m) = \theta_{P_k}(m) + 1$ for $m < k$. This implies $\tau_{\hat{C}_k}(m) = \frac{3}{2}\tau_{C_k}(m)$ and $\tau_{\hat{P}_k}(m) = 2\tau_{P_k}(m)$ for $m = 0, \ldots, k-1$.

Due to Lemma 4, looking for θ-optimal sets in hypergraphs we can restrict ourselves for weakly compressed sets only. Using Lemma 5, for a weakly compressed set $A \subseteq C_k^n$ we have

$$\begin{aligned} |\theta_{\hat{C}_k^n}(A)| &= \tfrac{3}{2}\,|\theta_{C_k^n}(A)|, \\ |\theta_{\hat{P}_k^n}(A)| &= 2\,|\theta_{P_k^n}(A)|. \end{aligned}$$

This implies that A is θ-optimal in $\hat{C}_k^n$ iff it is so for C_k^n, and similarly for $\hat{P}_k^n$ and P_k^n. A solution to the EIP for the graphs C_k^n and P_k^n is published in [9], where it is shown that these graphs admit θ-orders for $k = 3, 4, 5$. □

It turns out that the graphs C_k^n and P_k^n do not admit θ-orders for $n \geq 2$ and $k > 3$. As it follows from the proof of Theorem 5, it is also the case for the hypergraphs $\hat{C}_k^n$ and $\hat{P}_k^n$.

Another interesting phenomena is that the graphs C_k^n admit optimal E-orders for $k = 3, 4, 5$ and any $n \geq 1$ [9]. Also, the graphs P_k^n admit E-orders for any $k \geq 2$ and $n \geq 1$ [5]. However, as counterexamples show, none of the hypergraphs $\hat{C}_k^n$ and $\hat{P}_k^n$ admit optimal E-orders for $n \geq 2$. It would be interesting to find more examples of hypergraphs of the considered type, whose any cartesian power admits an E-order or a θ-order.

7 Conclusions and directions for further research

We have investigated the EIP problem on hypergraphs and their cartesian powers. The developed link to the poset extremal problems allowed us to apply some results from the Macaulay theory on posets to the EIP on hypergraphs. An important contribution is the construction of a hypergraph represented by a poset. This builds a missing peace in the theory. We also presented four graph families admitting nested solutions in the EIP.

The technique based on posets developed in Section 2 is applicable only to the hypergraphs with a bounded growth of the isoperimetric function. In Section 4 we extended the standard compression techniques for graphs to hypergraphs, which efficiently handles the cartesian powers of arbitrary hypergraphs with respect to both versions of the EIP. In Section 5 we presented an example of application of this technique to a special hypergraph family H_d^n for $d = 2, 3$. We conjecture that a similar result holds for every $d \geq 2$ and that the order H_d^n is the θ-order.

Concerning the version of the EIP dealing with the hyperedge-cut minimization, we were particularly concerned with finding a condition for a hypergraph in order to both versions of the EIP would be equivalent. This is known to be the case for regular graphs, but no regularity-based condition for hypergraphs that we tried, works. However, as we showed in Section 6, the hypergraph classes that we considered in Sections 2 and 5, do the job. It would be interesting to find further hypergraph classes for which both versions of the EIP are equivalent. In particular, we are interested to construct further graphs G such that any cartesian power of the hypergraph, obtained by adding to G a hyperedge including all its vertices, admits an E-order or a θ-order.

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